Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-04-17

Conceptual Review

Question 1. Qualitatively describe the following parametric surfaces.

(a) $\mathbf{r}(u,v) = \langle u+v, -u, v\rangle$

Plane

(b) $\mathbf{r}(u,v) = \langle u^2, -u, v\rangle$ for u const: vertical lines

for v const: horizontal parabolas

parabolic cylinder

(c) $\mathbf{r}(u,v) = \langle u\cos(2v), v, u\sin(2v)\rangle$ (you may see this surface again on an upcoming written HW)

There is a picture of this in §16.5 exercises; it is a "helical ramp"

Question 2': How to parametrize the curve $y=f(x)$: $\vec{r}(t) = \langle x, y\rangle = \langle t, f(t)\rangle$

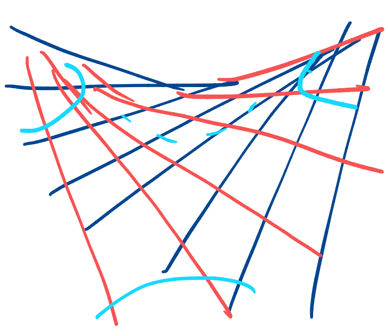
Question 2. What is one way you could parametrize the graph of a function $f(x,y)$, in other words the surface defined by $z = f(x,y)$?

$$\vec{r}(u,v) = \langle x, y, z\rangle = \langle u, v, f(u,v)\rangle$$

Question 3. Suppose that all of the u and v grid curves of a parametric surface $\mathbf{r}(u,v)$ are lines. Does it follow that the parametric surface must be a plane?

No, consider $\vec{r}(u,v) = \langle u, v, uv\rangle$

This is $z = xy$, a hyperbolic paraboloid



Problems

Problem 1. Find a normal vector to the plane $\mathbf{r}(u, v) = \langle \overset{x}{u + v}, \overset{y}{u - v}, \overset{z}{2u} \rangle$ by

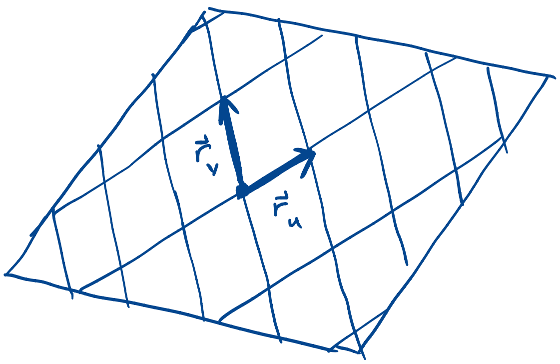
(a) finding a Cartesian equation for the plane (i.e. a single equation of x, y, z) and reading off a normal vector from that.

$$(u+v) + (u-v) - 2u = 0$$

$$x + y - z = 0 \quad \text{so a normal vec is: } \langle 1, 1, -1 \rangle$$

(b) using the $\mathbf{r}_u \times \mathbf{r}_v$ method.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = \langle 2, 2, -2 \rangle.$$

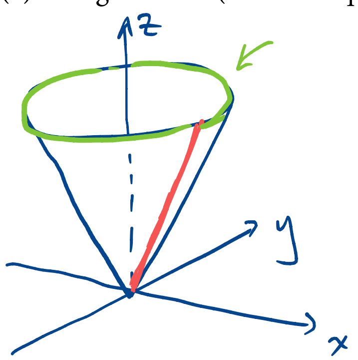


$$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$$

Problem 2. Consider the portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$. Compute its surface area

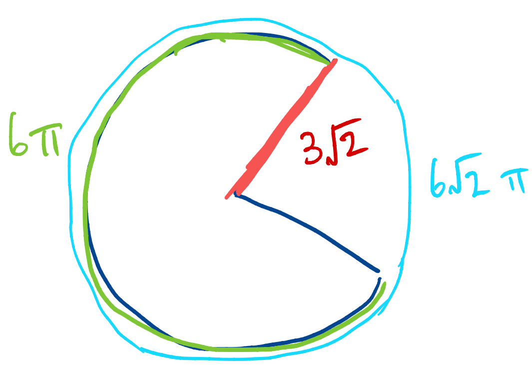
(a) using calculus (i.e. techniques from §16.5).

$$|\vec{r}_u \times \vec{r}_v| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = \sqrt{2}.$$



$$SA = \iint_{\substack{uv \\ \text{region}}} \underbrace{|\vec{r}_u \times \vec{r}_v| \, du\, dv}_{dS} = \boxed{\sqrt{2} \cdot 9\pi}$$

(b) without using calculus. Hint: cut a slit in the cone and unfold it into a “pacman” shape.



$$\text{Area} = 18\pi \cdot \frac{6\cancel{\pi}}{\cancel{6}\sqrt{2}\,\cancel{\pi}} = \boxed{9\pi\sqrt{2}}$$